

Untwisting 3-strand torus knots

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ABSTRACT

We prove that the signature bound for the topological 4-genus of 3-strand torus knots is sharp, using McCoy’s twisting method. We also show that the bound is off by at most 1 for 4-strand and 6-strand torus knots, and improve the upper bound on the asymptotic ratio between the topological 4-genus and the Seifert genus of torus knots from $2/3$ to $14/27$.

1. Introduction

The braid group on 3-strand B_3 is generated by two elements a, b satisfying the braid relation $aba = bab$. In this note, we are interested in the natural closure of the positive braid $(ab)^n$ in S^3 , known as torus link of type $T(3, n)$. Whenever $n \in \mathbb{N}$ is a multiple of 3, the link $T(3, n)$ has three components; otherwise it is a knot. The topological 4-genus $g_t(K)$ of a knot $K \subset S^3$ is defined to be the minimal genus among all surfaces $\Sigma \subset D^4$, embedded in a locally flat way, with boundary $\partial\Sigma = K$. As with the smooth version of the 4-genus invariant, the topological 4-genus of knots K is bounded below by the signature invariant [11]: $g_t(K) \geq |\sigma(K)|/2$. The same lower bound holds with the signature invariant replaced by the maximum value of the Levine–Tristram signature function outside of the set of roots of the Alexander polynomial $\Delta_K(t)$ of K

$$\widehat{\sigma}(K) := \max_{\omega \in S^1 \setminus \Delta_K^{-1}(0)} |\sigma_\omega(K)|.$$

THEOREM 1. *Let $n \geq 4$ be a natural number not divisible by 3. Then*

$$g_t(T(3, n)) = \frac{\widehat{\sigma}(T(3, n))}{2} = \left\lceil \frac{2n}{3} \right\rceil.$$

We believe that the equality $g_t = \widehat{\sigma}/2$ holds for a much larger class of torus knots, possibly for all. This can be seen as a topological counterpart of the local Thom conjecture, which states that the smooth 4-genus g_s of torus knots coincides with their Seifert genus [6, 12, 14]. Unlike in the smooth case, where the hard part is finding suitable lower bounds, the difficulty in the topological case is figuring out genus-minimising surfaces (see [2, 13] for first attempts in this direction). We will not see any of these surfaces. Rather, we will find a precise upper bound for the topological 4-genus via an operation called null-homologous twisting, which has recently received some attention [5, 8–10]. A null-homologous twist is an operation on oriented links that inserts a full twist into an even number $2m$ of parallel strands, m of which point upwards, and m of which point downwards (see, for example, Figure 3). Throughout this paper,

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we will use the term *twist* for a null-homologous twist. The case of two strands corresponds to a simple crossing change. For a knot K , we define the *untwisting number* $t(K)$ to be the minimal number of twists needed to transform K into the trivial knot, as in [5]. Relying on Freedman's disc theorem [3], McCoy proved that the untwisting number is an upper bound for the topological 4-genus of knots [9]. This is the tool we use to construct the genus-minimising surfaces in Theorem 1.

Let us take another look at the resemblance of the smooth and topological setting. Writing s and u for the Rasmussen invariant and unknotting number, respectively, it follows from the (smooth) local Thom conjecture that the inequalities

$$s(K)/2 \leq g_s(K) \leq u(K),$$

which hold for all knots K , become equalities for all torus knots:

$$s(T(p, q))/2 = g_s(T(p, q)) = u(T(p, q)).$$

We show that in the topological setting, in striking analogy, the inequalities

$$\widehat{\sigma}(K)/2 \leq g_t(K) \leq t(K),$$

which hold for all knots K , become equalities for all 3-strand torus knots:

$$\widehat{\sigma}(T(3, n))/2 = t(T(3, n)) = g_t(T(3, n)).$$

Thus in the topological setting, the untwisting number apparently takes the place that the unknotting number has in the smooth setting.

Untwisting might very well lead to the equality $g_t = \widehat{\sigma}/2 = t$ for all torus knots. For the time being, we show that the equality is off by at most 1 for torus knots with four and six strands.

PROPOSITION 2. *For all odd natural numbers $n \geq 3$,*

$$n \leq \frac{\widehat{\sigma}(T(4, n))}{2} \leq g_t(T(4, n)) \leq t(T(4, n)) \leq \frac{2}{3}g(T(4, n)) + 2 = n + 1.$$

Moreover, for all natural numbers $n \geq 5$ coprime to 6,

$$\frac{3n+1}{2} \leq \frac{\widehat{\sigma}(T(6, n))}{2} \leq g_t(T(6, n)) \leq t(T(6, n)) \leq \frac{3}{5}g(T(6, n)) + 3 = \frac{3n+3}{2}.$$

McCoy also developed an induction scheme that allows him to estimate the asymptotic ratio between the topological 4-genus and the Seifert genus of torus knots [9]:

$$\limsup_{p, q \rightarrow \infty} \frac{g_t(T(p, q))}{g(T(p, q))} \leq \frac{2}{3}.$$

THEOREM 3.

$$\limsup_{p, q \rightarrow \infty} \frac{g_t(T(p, q))}{g(T(p, q))} \leq \frac{14}{27} \approx 0.519.$$

The proof of Theorem 1 uses a calculus for positive 3-braids introduced in [1], which we present in the next section. Sections 3 and 4 contain the proofs of Theorems 1 and 2. The latter follows from the former by untwisting torus knots on four and six strands via torus knots on three strands. Theorem 3 follows from McCoy's induction scheme, which we briefly review in the last section.

2. A calculus for positive 3-braids

Let k_1, k_2, \dots, k_n be strictly positive integers. The positive braid

$$[k_1, k_2, \dots, k_n] := a^{k_1} b a^{k_2} b \dots a^{k_n} b \in B_3$$

defines a link $L[k_1, k_2, \dots, k_n]$, via its closure. For example, the torus link of type $T(3, n)$ can be written as $L[1, 1, \dots, 1]$, where the number 1 appears n times. This notation is far from unique. The full twist on three strands can be written as

$$[1, 1, 1] = ababab = aabaab = [2, 2].$$

The double full twist can be written as

$$aab(abaaba)aab = aaabaaabaab = [3, 3, 3].$$

In the first equality, we used the fact that the full twist $abaaba \in B_3$ commutes with all 3-braids. Adding another full twist to this, we obtain the following representative for the triple full twist:

$$aaabaaab(abaaba)aaab = aaabaaabaabaabaab = [3, 4, 3, 4].$$

From here, we see that the operation

$$[\dots, x, y, \dots] \rightarrow [\dots, x + 1, 3, y + 1, \dots]$$

corresponds to adding a full twist to a given positive braid on three strands. With this combinatorial calculus, we obtain the following family of positive braid presentations for iterated full twists on three strands.

LEMMA 4. For all $k \in \mathbb{N}$:

- (1) $T(3, 6k + 9) = L[3, 5^k, 4, 3, 5^k, 4]$,
- (2) $T(3, 6k + 12) = L[3, 5^{k+1}, 3, 4, 5^k, 4]$,

where 5^k stands for a sequence $5, \dots, 5$ of length k .

The proof is by induction on k , starting at zero. A repeated application of the above move yields the desired sequence of presentations for increasing powers of the full twist:

$$[3, 5, 3, 4, 4], [3, 5, 4, 3, 5, 4], [3, 5, 5, 3, 4, 5, 4], [3, 5, 5, 4, 3, 5, 5, 4], \dots$$

3. Untwisting torus knots with three strands

LEMMA 5. For all $k \in \mathbb{N}$:

- (1) $t(T(3, 3k + 4)) \leq 2k + 3$,
- (2) $t(T(3, 3k + 5)) \leq 2k + 4$.

Proof. The two statements are obviously true for $k = 0$, since the knots $T(3, 4)$ and $T(3, 5)$ can be unknotted by three and four crossing changes, respectively. Moreover, for all $k \in \mathbb{N}$, the two knots $T(3, 3k + 4)$ and $T(3, 3k + 5)$ are related by a single crossing change, so we only need to prove (1). We will do so by considering the three special cases $T(3, 7)$, $T(3, 10)$, $T(3, 13)$ separately, and then the two families $T(3, 6k + 16)$, $T(3, 6k + 19)$.

The key observation is that the 2-braids $abbaabba$ and bb are related by a sequence of two twists, as shown in Figure 1. Here the first arrow stands for a twist on four strands, while the second arrow is a simple crossing change.

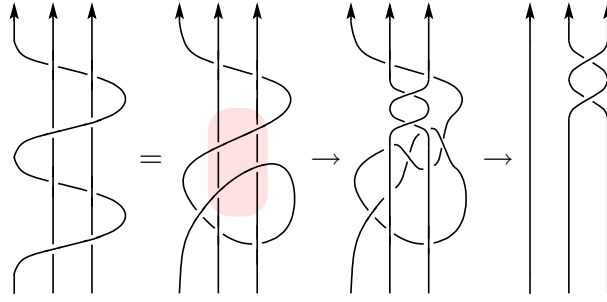


FIGURE 1 (colour online). Sequence of two twists. The first twist is on four strands, marked in red. The second twist is a crossing changes, untying the trefoil summand in the third drawing.

As a consequence, the double full twist on three strands,

$$(ab)^6 = abbaabbabbbb,$$

is related to the braid b^6 (and also to a^6) by a sequence of two twists. For the first knot, $T(3, 7)$, we turn the braid $(ab)^7$ into a^7b by two twists, and into ab by another three crossing changes, thus showing $t(T(3, 7)) \leq 5$. In order to deal with the other two knots, we use the notation $A = a^{-1}$, $B = b^{-1}$. We write

$$(ab)^{10} = (ab)^{12}BABA = (ab)^{12}ABAA = AB(ab)^{12}AA,$$

which transforms into $ABb^6a^6AA = Ab^5a^4$ by a sequence of four twists, and into Aba^2 by another three twists. The closure of the last braid is the trivial knot; this shows $t(T(3, 10)) \leq 7$. For the knot $T(3, 13)$, we observe that the braid

$$(AB)^5 = (AB)^6ba = A^3BA^3BA^3Bba = A^3BA^3BA^2$$

represents the torus knot $T(3, -5)$. Therefore, we can write

$$(ab)^{13} = (ab)^{18}A^3BA^3BA^2 = A^3(ab)^{12}BA^3(ab)^6BA^2,$$

which transforms into $A^3a^6b^6BA^3a^6BA^2 = a^3b^5a^3BA^2$ by a sequence of six twists, and into $a^3baBA^2 = a^2bA$ by another three twists. This shows $t(T(3, 13)) \leq 9$.

We now turn to the family of torus knots $T(3, 6k + 16)$. Using again $A = a^{-1}$, $B = b^{-1}$, we write

$$T(3, 6k + 16) = L[ab(ab)^{6(2k+4)}(BA)^{6k+9}].$$

By Lemma 4, we have

$$(BA)^{6k+9} = A^3B(A^5B)^kA^4BA^3B(A^5B)^kA^4B,$$

which contains precisely $2k + 4$ pure powers of A . We slide one double full twist $(ab)^6$ to the right of each power of A and transform it into a^6 by a sequence of $2(2k + 4)$ twists, in total. This leaves us with the braid

$$aba^3B(aB)^ka^2Ba^3B(aB)^ka^2B.$$

Sliding the half-twist aba from the left to the middle yields

$$b^2A(bA)^kb^2abaBa^3B(aB)^ka^2B.$$

Then we transform the middle part $b^2abaBa^3B = b^3a^4B$ into the empty braid by three crossing changes. What remains is the braid b^2Aa^2B , whose closure is the trivial knot. Therefore $t(T(3, 6k + 16)) \leq 2(2k + 4) + 3 = 4k + 11$, in accordance with statement (1).

The second family, $T(3, 6k + 19)$, works in complete analogy, using the expression

$$T(3, 6k + 19) = L[ab(ab)^{6(2k+5)}(BA)^{6k+12}]$$

and

$$(BA)^{6k+12} = A^3 B(A^5 B)^{k+1} A^3 B A^4 B(A^5 B)^k A^4 B.$$

The resulting intermediate braid, after a sequence of $2(2k + 5)$ twists, is

$$aba^3 B(aB)^{k+1} a^3 B a^2 B(aB)^k a^2 B = b^2 A(bA)^{k+1} b^3 A b^2 a b a B(aB)^k a^2 B.$$

Again, the middle part $b^3 A b^2 a b$ transform into the empty braid by three crossing changes. The remaining braid is $b^2 a B$, whose closure is the trivial knot. This shows $t(T(3, 6k + 19)) \leq 2(2k + 5) + 3 = 4k + 13$, in accordance with statement (1). \square

As mentioned before, the inequalities

$$\widehat{\sigma}(K)/2 \leq g_t(K) \leq t(K)$$

hold for all knots K . To complete the proof of Theorem 1, we estimate the maximal Levine–Tristram signature $\widehat{\sigma}$ for the 3-strand torus knots. Let K be a knot and consider the jump function

$$\delta_K(x) = \lim_{s \rightarrow x^+} \sigma_{e^{2\pi i s}}(K) - \lim_{s \rightarrow x^-} \sigma_{e^{2\pi i s}}(K).$$

Litherland (see the comments after [7, Proposition 1]) notes that the discontinuities of the Levine–Tristram signature of the $T(p, q)$ torus knot occur precisely at the $x \in (0, 1)$ satisfying $pqx \in \mathbb{Z}$ but $px \notin \mathbb{Z}$ and $qx \notin \mathbb{Z}$. For x a discontinuity, let $pqx = pa + qb$ with $0 < a < q$. Litherland also shows that $\delta(x) = +2$ if $b < 0$ and $\delta(x) = -2$ if $b > 0$. If x is the minimal discontinuity satisfying $x \geq \frac{1}{2}$, then $\widehat{\sigma}(K) \geq \sigma(K) + \delta_K(x)$. We distinguish the following cases.

$T(3, 3k + 4)$ for k even: The periodicity of the ordinary signature, see [4, Theorem 5.2], implies that $\widehat{\sigma}(T(3, 3k + 4)) \geq \sigma(T(3, 3k + 4)) = \sigma(T(3, 4)) + 4k = 6 + 4k$.

$T(3, 3k + 5)$ for k even: Similarly to the previous case, $\widehat{\sigma}(T(3, 3k + 5)) \geq 8 + 4k$.

$T(3, 3k + 4) = T(3, 6l + 7)$ for $k = 2l + 1$ odd: The minimal $x \geq \frac{1}{2}$ satisfying $pqx \in \mathbb{Z}, px \notin \mathbb{Z}, qx \notin \mathbb{Z}$ is $x = (9l + 11)/(3(6l + 7))$. Note that $pqx = 9l + 11 = 3(5l + 6) + (6l + 7)(-1)$, which implies that $\delta(x) = +2$. The estimate $\widehat{\sigma} \geq \sigma + \delta(x)$ and the periodicity of the ordinary signature imply $\widehat{\sigma}(T(3, 3k + 4)) \geq \sigma(T(3, 3k + 4)) + 2 = \sigma(T(3, 3(k + 1) + 1)) = 4(k + 1) + 2 = 4k + 6$.

$T(3, 3k + 5) = T(3, 6l + 8)$ for $k = 2l + 1$ odd: Similar to the previous case, the minimal discontinuity $x \geq \frac{1}{2}$ is $x = (9l + 13)/(3(6l + 8))$. From $pqx = 9l + 13 = 3(5l + 7) + (6l + 8)(-1)$ it follows that $\delta(x) = 2$ and then $\widehat{\sigma}(T(3, 3k + 4)) \geq 4k + 8$.

Therefore, for all $k \in \mathbb{N}$,

$$\widehat{\sigma}(T(3, 3k + 4)) \geq 4k + 6, \quad \widehat{\sigma}(T(3, 3k + 5)) \geq 4k + 8,$$

and Theorem 1 now follows.

4. Untwisting torus knots with four and six strands

To prove Proposition 2, we essentially untwist torus knots with four and six strands to torus knots on three strands, and conclude using Theorem 1.

By a similar calculation as was presented at the end of the previous section for 3-strand torus knots, one may prove that $\widehat{\sigma}(T(4, n)) \geq 2n$ and $\widehat{\sigma}(T(6, n)) \geq 3n + 1$. For this, consider the signature (for $T(4, 4k + 3)$ and $T(6, k + 5)$), or the Levine–Tristram signature after the

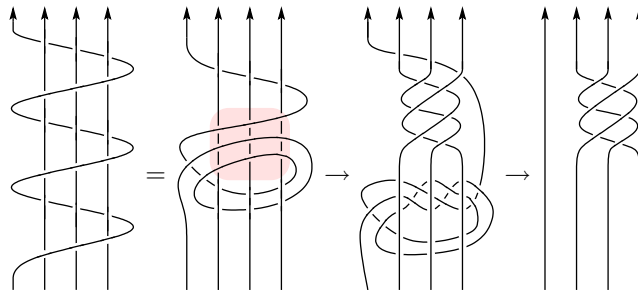


FIGURE 2 (colour online). Sequence of four twists. The first twist is on six strands, marked in red. The other three twists are crossing changes, untying the $T(3, 4)$ summand in the third drawing.

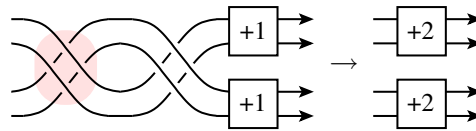


FIGURE 3 (colour online). Untwisting a full twist on four strands, marked in red. The numbers $+1(+2)$ stand for a (double) positive full twist.

first jump after $1/2$ (for $T(4, 4k + 1)$), or after the second jump (for $T(6, 6k + 1)$). We note (without proof) that the stated inequalities for $\hat{\sigma}$ are in fact equalities.

Now, let us show that $t(T(4, n)) \leq n + 1$. Denote the standard Artin generators of the braid groups B_4 by a, b, c . The crucial move is to transform three full twists on four strands, $(abc)^{12}$, into four full twists on three strands, $(bc)^{12}$ (or $(ab)^{12}$), by four twist operations. This can be seen by composing the braids in Figure 2 with $(bc)^9$.

Hence, for $n = 12k + \varepsilon$, with $\varepsilon \in \{\pm 1, \pm 3, \pm 5\}$, one may change $T(4, n) = L[(abc)^n]$ by $4k$ twists into $L[(ab)^{12k}(abc)^\varepsilon] =: K_\varepsilon$. We now consider the possible values of ε one by one, showing $t(T(4, n)) \leq n + 1$ in each case, thus completing the proof of the first part of Proposition 2.

- K_1 is in fact $T(3, 12k + 1)$, which may be untwisted by $8k + 1$ twists, as established previously. In total, $t(T(4, n)) = n$.
- Similarly $K_{-1} = T(3, 12k - 1)$, which may be untwisted by $8k$ twists, resulting in $t(T(4, n)) \leq n + 1$.
- $K_3 = L[(ab)^{12k}(abc)^3] = L[(ab)^{12k}aba^2b^2ab] = T(3, 12k + 4)$, which may be untwisted by $8k + 3$ twists. In total $t(T(4, n)) = n$.
- Similarly, $K_{-3} = T(3, 12k - 4)$, which may be untwisted by $8k - 2$ twists, giving a total of $t(T(4, n)) \leq n + 1$.
- K_5 can be transformed into K_1 by a twist on two strands and four crossing changes (cf. Figure 3), in total $t(T(4, n)) \leq n + 1$.
- Similarly, K_{-5} can be transformed into $T(3, 12(k - 1) + 1)$ by five twists, resulting in $t(T(4, n)) \leq n + 1$.

This concludes the proof of the first half of Proposition 2.

To show $t(T(6, n)) \leq (3n + 3)/2$, denote the standard Artin generators of B_6 by a, b, c, d, e , respectively. The full twist $(abcde)^6$ on six strands may be transformed by a single twist into $(ab)^6(de)^6$, see Figure 3 for the analogous operation on four instead of six strands. Applying this k times to $T(6, 6k \pm 1)$ yields the connected sum of two copies of $T(3, 6k \pm 1)$, which is finished off using Theorem 1. Summing up, the second half of Proposition 2 follows.

5. Asymptotic genus ratio

The key point in McCoy's induction scheme is that a positive full twist in a braid with $2n$ strands can be transformed into two parallel copies of positive double full twists in n strands, with a single twist operation (see [9, Lemma 13]). This is shown in Figure 3, for $n = 2$, and was used in the previous section for $n = 2$ and $n = 3$. Similarly, a single twist operation transforms the torus knot $T(2n, 2n + 1)$ into the connected sum of two copies of the torus knot $T(n, 2n + 1)$. This is seen by adding the word $\sigma_1\sigma_2 \dots \sigma_{2n-1}$ to both braids in the same figure.

When iterating this operation on successive powers of two, one gets $2/3$ as an upper bound for the asymptotic ratio g_t/g for torus knots with increasing parameters. We will apply the same procedure, starting from braids with three strands, successively multiplying the strand number by 2:

- (1) $T(6, 7)$ transforms into the disjoint union of two copies of $T(3, 7)$ by one twist;
- (2) $T(12, 13)$ transforms into the disjoint union of two copies of $T(6, 13)$ by one twist, then into the disjoint union of four copies of $T(3, 13)$ by 4 more twists;
- (3) $T(24, 25)$ transforms into the disjoint union of two copies of $T(12, 25)$ by one twist, then into the disjoint union of eight copies of $T(3, 25)$ by $4(1 + 4) = 20$ more twists,
- (4) $T(3 \cdot 2^k, 3 \cdot 2^k + 1)$ transforms into the disjoint union of 2^k copies of $T(3, 3 \cdot 2^k + 1)$ by a total number of $1 + 4 + 16 + \dots + 4^{k-1} = 1/3 \cdot (4^k - 1)$ twists.

By Theorem 1, the untwisting number of $T(3, 3 \cdot 2^k + 1)$ is of the order

$$2/3 \cdot (3 \cdot 2^k) = 2^{k+1}.$$

We conclude that the untwisting number of $T(3 \cdot 2^k, 3 \cdot 2^k + 1)$ is bounded above by an expression of the order

$$1/3 \cdot (4^k - 1) + 2^k \cdot 2^{k+1} \approx (1/3 + 2) \cdot 4^k,$$

while its Seifert genus is of the order

$$1/2 \cdot (3 \cdot 2^k)^2 = 9/2 \cdot 4^k,$$

by the well-known genus formula $g(T(p, q)) = 1/2 \cdot (p - 1)(q - 1)$. In summary,

$$\limsup_{k \rightarrow \infty} \frac{g_t(T(3 \cdot 2^k, 3 \cdot 2^k + 1))}{g(T(3 \cdot 2^k, 3 \cdot 2^k + 1))} \leq \frac{1/3 + 2}{9/2} = \frac{14}{27}.$$

The existence of the more general upper bound,

$$\limsup_{p, q \rightarrow \infty} \frac{g_t(T(p, q))}{g(T(p, q))} \leq \frac{14}{27},$$

follows from a general principle on subadditive functions, see the proof of [9, Theorem 5], or the paragraph preceding [2, Theorem 2]. We are left with the strong belief that the ratio tends to $1/2$, in accordance with the asymptotic behaviour of the signature invariant:

$$\lim_{p, q \rightarrow \infty} \frac{\sigma(T(p, q))}{2g(T(p, q))} = \frac{1}{2}.$$

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